Assignment 5.

This homework is due *Thursday*, October 8.

There are total 32 points in this assignment. 27 points is considered 100%. If you go over 27 points, you will get over 100% (up to 115%) for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.1–3.2 in Bartle–Sherbert.

(1) Reminder. Recall definition of a sequence in $\mathbb{R}$ converging to an $x \in \mathbb{R}$: Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ converges to x if $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall n > K, |x - x_n| < \varepsilon$.

Below you can find (erroneous!) "definitions" of a sequence converging to x. In each case describe, exactly which sequences are "converging to x" according to that "definition".

(a) [2pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if $\forall \varepsilon > 0\ \forall K \in \mathbb{N}\ \forall n > K, |x - x_n| < \varepsilon$.
(*If you are confused at this point, think of the problem this way: suppose for some sequence (x_n) and a number $x \in \mathbb{R}$ you know that statement (a) is true. What can you say about about (x_n)?*)

(b) [2pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if $\exists K \in \mathbb{N}\ \forall \varepsilon > 0\ \forall n > K, |x - x_n| < \varepsilon$.

(c) [2pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ converges to x if $\exists \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall n > K, |x - x_n| < \varepsilon$.

(d) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$, let $x \in \mathbb{R}$. $X = (x_n)$ "converges to x" if $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \exists n > K, |x - x_n| < \varepsilon$.

(2) [3pt] (Exercise 3.2.7) If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 (Arithmetic properties of limit, "$\lim XY = \lim X \cdot \lim Y$") *cannot* be used.

(3) (a) [2pt] (Theorem 3.2.3) Let $X = (x_n)$ and $Y = (y_n)$ be sequences in $\mathbb{R}$ converging to x and y, respectively. Prove that $X - Y$ converges to $x - y$.

(b) [1pt] (Exercise 3.2.3) Show that if X and Y are sequences in $\mathbb{R}$ such that X and $X + Y$ converge, then Y converges.

(c) [2pt] (Exercise 3.2.2b) Give an example of two sequences X, Y in $\mathbb{R}$ such that XY converges, while X and Y do not.

— see next page —

(4) [4pt] Determine the following limits (or establish they do not exist):

(a) $\lim\limits_{n\to\infty} \frac{2n^2-1}{1000n+100000}$ (*Hint:* show that the sequence is not bounded),

(b) $\lim\limits_{n\to\infty} \frac{2\sqrt{n^2+1}-10}{1000n+100000}$,

(c) $\lim\limits_{n\to\infty} \frac{2n^2-1}{1000\sqrt[5]{n^{11}+12}-100000}$.

(5) Find limits using Squeeze Theorem:

(a) [2pt] $\lim\limits_{n\to\infty} \frac{n^2+2015n(\sin n+3\cos n^7)-1}{2n^2-\cos(3n^2+1)}$,

(b) [2pt] $\lim\limits_{n\to\infty} \sqrt{n^2+\cos(2014n+1)} - \sqrt{n^2-\sin(n^3-1)}$. (*Hint:* Once you get rid of sin and cos, multiply and divide by the conjugate, $\sqrt{\ } + \sqrt{\ }$.)

(6) (a) [3pt] (Example 3.1.11d Prove that $n^{1/n} \to 1$ $(n \to \infty)$.

(b) [2pt] (3.2.14a) Use Squeeze theorem to find limit of the sequence (n^{1/n^2}).

(7) [2pt] (3.2.8) Find a mistake in the following argument.
"Find $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n$ as shown below:

$$\begin{aligned}
\lim_{n\to\infty}(1+\tfrac{1}{n})^n &= \lim_{n\to\infty}(1+\tfrac{1}{n})\cdot(1+\tfrac{1}{n})\cdots(1+\tfrac{1}{n}) \\
&= \lim_{n\to\infty}(1+\tfrac{1}{n})\cdot\lim_{n\to\infty}(1+\tfrac{1}{n})\cdots\lim_{n\to\infty}(1+\tfrac{1}{n}) \\
&= \left(\lim_{n\to\infty}(1+\tfrac{1}{n})\right)^n = 1^n = 1.
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = 1$."

Comment. To reiterate, the argument above is erroneous and the obtained value of the limit is wrong, too.